Mean-Risk and Stochastic Dominance Efficient Sets Under Comovements

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Abstract

In this paper we exploit the relationship between mean-risk and stochastic dominance efficient sets in order to derive optimal portfolio strategies that can be superior to the mean-variance benchmark under comovements episodes of the market. To do this we decompose the risk in a portfolio in risk due to the likelihood of comovements and risk due to the allocation of weights. In this way we are able to discern whether a portfolio defined by an optimal allocation of weights is optimal under comovements episodes, or there are other allocations better suited to minimize risk in this regime. To implement this we develop a novel test statistic for testing stochastic dominance that, in contrast to most test statistics in the literature, makes allowance for testing orders of dominance higher than zero and for some types of dependence between portfolios. This test is extended to conditional stochastic dominance by using a simple reformulation of the test statistic and the asymptotic theory. These results are illustrated in the empirical application where we stress the importance of the marginal distribution of the assets in the portfolio and the likelihood of comovements for the mean-variance efficient portfolio to be stochastically superior to alternative mean-risk efficient sets.

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1 Introduction

It was Markowitz (1952) who formalized the concept of portfolio diversification by showing that investors should choose assets as if they care only about the mean and variance of the returns on an investment portfolio and therefore should penalize equally departures from expected wealth in both sides. Alternatively, Roy (1952) developed the concept of safety-first portfolios where investors' aim consisted on minimizing the likelihood of a dread event, this identified with an outcome in the tail of the distribution of portfolio returns. Roy, as Markowitz, also confined himself to distributions defined by the first two statistical moments. Following this alternative interpretation of risk Markowitz (1959) proposed the semivariance, risk measure that only focused on deviations of the return on the portfolio below a target return determined by the expected return on the investment or the return on the risk-free asset.

Hogan and Warren (1974), Bawa (1975), Arzac and Bawa (1977), and Bawa and Lindenberg (1977) continued on the idea of risk based on dread events introduced by Roy and proposed different risk measures based on penalizing the chance of these events. Thus, building on Roy's (1952) formulation of risk and extending the semivariance of Markowitz (1959) these authors introduced lower partial moments (LPM) of the distribution of returns to describe risk. Bawa ((1975), (1976), (1978)) provided some microeconomic foundations to these risk measures by introducing a family of utility functions consistent with them that described the preferences of downside-risk averse investors. These functions take this form:

$$U(R_p; q, \tau) = R_p - k(\tau - R_p)^q I(R_p \leq \tau), \tag{1}$$

where R_p is the return on a portfolio P; τ is the threshold denoting the target return; k a scale parameter, $I(\cdot)$ an indicator function that takes the value one if $R_p \leq \tau$ and zero otherwise, and q the degree of risk aversion of the investor.

Bawa and Lindenberg (1977) and Harlow and Rao (1989) showed that the optimal portfolio choice of downside-risk averse investors is the solution of the following equation,

$$\min_{w} LPM_q(\tau) = \int_{-\infty}^{\tau} (\tau - R_p)^q dF(R_p), \tag{2}$$

where $R_p = \sum_{j=1}^{m} w_j R_j$, with m the number of assets, and subject to $0 \leq w_j \leq 1$, for all j, $\sum_{j=1}^{m} w_j = 1$ and $\sum_{j=1}^{m} w_j E[R_j] \leq \mu(P)$ where $\mu(P)$ denotes the expected return level. The curve of all efficient portfolios is called mean-risk efficient set, in contrast to the mean-variance efficient set derived from minimizing the variance.

Fishburn (1977) also explores these downside-risk measures and the corresponding mean-risk models and shows the close connection between the concepts of mean-risk dominance and of stochastic dominance. In particular this author shows that the efficient sets obtained from minimizing LPM_q measures are a subset of the different efficient first, second and third stochastic dominance sets. This result brought attention to hypothesis tests for stochastic dominance as McFadden (1989), Kaur, Rao and Singh (1994), Anderson (1996) or Davidson and Duclos (2000). Another interesting implication of this concept is its dominance over the mean-variance model under general conditions. For example, it is well known that there can exist mean-variance portfolio allocations that are dominated in the second order stochastic dominance sense for all risk-averse agents, see Gotoh and Konno (2000) or Manganelli (2007). This can be the case if the multivariate distribution of returns on the portfolio is not gaussian or the utility function employed to model individuals' preferences is not quadratic. The normality assumption is often used because it leads to tractable results, however empirical studies during the last forty years have been consistently rejecting this hypothesis and pointing towards heavy tailed distributions, see Fama (1965). This stylized fact has gained further popularity during the last decade where more sophisticated statistical and probabilistic techniques have been developed to study heavy tails and extreme events, see Embrechts, Klüppelberg and Mikosch (1997) for an excellent review.

An important factor influencing mean-risk and stochastic dominance efficient sets in portfolio analysis is the phenomenon of comovements. This concept, widely associated with an increase in the correlation between assets in crises episodes or with an increase in the joint dependence in the tails of the multivariate distribution of the returns on a portfolio, has gained much attention in the financial literature since the development of downside-risk measures, see previous references, and the implementation of risk management measures as Value-at-Risk, see Basel Accord (1996). Since the vocation of these methodologies is to minimize the likelihood of events corresponding to negative returns extra caution to the presence of comovements in crises episodes should be exercised. This phenomenon also influences the construction of optimal portfolios whose composition can be heavily affected by the market regime. Also interesting is the comparison between two portfolios that are unconditionally non-dominated by each other for any order but such that one stochastically dominates the other under distress episodes of the market, identified with comovements periods.

To the best of our knowledge the concept of stochastic dominance conditional on being on a comovements regime and the decomposition of downside-risk measures in terms of the probability of occurrence of this phenomenon have not been studied yet. Neither have been hypothesis tests for stochastic and conditional stochastic dominance for orders of dominance higher than zero and that make allowance for dependence between portfolios. Thus, this paper complements and extends the influential work of Barret and Donald (2003) in three directions. First, we allow for dependence between portfolios, second, the relevant distribution functions can have infinite supports, and finally we extend the concept of stochastic dominance to conditional stochastic dominance.

The main aim of the paper is to exploit the relationship between mean-risk and stochastic dominance efficient sets in order to derive optimal portfolio strategies that can be superior to the mean-variance benchmark under some conditions. We concentrate, in particular, in deriving the efficient portfolio frontier for risk-neutral and risk-averse investors under distress episodes of the market described by the presence of comovements between the assets comprising an investment portfolio. To do this we need first to find the contribution to risk in a portfolio due to the likelihood of comovements between the assets and the contribution to risk due to the allocation of weights. In this way we will be able to discern if an optimal portfolio defined by an optimal allocation of weights is so under comovements episodes, or there are other allocations better suited to minimize risk under this scenario.

To make these concepts operational we provide a statistical mechanism to sort portfolios that is based on testing for stochastic dominance, and where the order of dominance will depend on the degree of investors' risk-aversion. We develop a novel test statistic that overcomes the existing burdens for testing the hypothesis of dominance for higher orders than zero and that makes allowance for some types of dependence between portfolios. This test is extended to conditional stochastic dominance by using a simple reformulation of the test statistic and the asymptotic theory.

The paper is structured as follows. Section 2 introduces the decomposition of LPM_q measures in terms of the level of comovements between the assets in the portfolio, the definition of conditional stochastic dominance and its relation with mean-risk efficiency under comovements. Section 3 introduces different estimators of the LPM risk measures, derives the relevant hypothesis tests for testing stochastic dominance and conditional stochastic dominance, and the asymptotic theory. In Section 4 we carry out a Monte Carlo simulation experiment to study the power curve and other appealing features of the different test statistics proposed under different scenarios. Section 5 studies the mean-variance efficient set and compares it with other mean-risk efficient frontiers via stochastic and conditional stochastic dominance for real data from US and UK financial markets. Finally Section 6 concludes with the main findings of the paper. Proofs are gathered in a mathematical appendix.

2 Decomposition of LPM Measures in terms of Comovements

We will start by introducing our definition of comovements. This phenomenon represents the collapse of the financial system, in our case identified with a state of the market where every return on the portfolio of interest is below a threshold u, and will be measured by $P\{R_1 \leq u, \ldots, R_m \leq u\} := \alpha(u)$. We will assume first that u is fixed, and later this will be relaxed to let the threshold go to the left end point of the relevant distributions and use results from extreme value theory.

The relevant risk measures in this context for portfolio optimization are

$$LPM_q(\tau) = \int_{-\infty}^{\tau} (\tau - R_p)^q dF(R_p), \tag{3}$$

with $F(R_p) := P\{R_p \leq x\}$, and

$$LPM_{q,u}(\tau) = \int_{-\infty}^{\tau} (\tau - R_p)^q dF_u(R_p), \tag{4}$$

with $F_u(R_p) := P\{R_p \leq x | R_1 \leq u, \ldots, R_m \leq u\}$ for comovements episodes.

After some simple algebra these formulas can be written as

$$LPM_q(\tau) = E[(\tau - R_p)^q | R_p \leq \tau] LPM_0(\tau), \tag{5}$$

and

$$LPM_{q,u}(\tau) = E[(\tau - R_p)^q | R_p \leq \tau, R_1 \leq u, \ldots, R_m \leq u\}] LPM_{0,u}(\tau). \tag{6}$$

The proof of the second result is obtained by noting that $LPM_{0,u}(x) = LPM_{0,u}(\tau) P\{R_p \leq x | R_p \leq \tau, R_1 \leq u, \ldots, R_m \leq u\}$, for all $x \leq \tau$.

These downside-risk measures describe the behavior and preferences of risk-neutral investors for $q = 0$ and those of risk-averse investors for $q > 0$. Although the case of risk-neutrality is not very interesting in practice since investors are assumed to expect a reward for the risk they are facing we will confine ourselves first to study the decomposition of LPM_0 for illustration purposes. By using the conditional probability theorem we obtain that

$$LPM_0(\tau) = \alpha(u) LPM_{0,u}(\tau) + (1 - \alpha(u)) LPM_{0,nu}(\tau), \tag{7}$$

with $LPM_{0,nu}(\tau) := P\{R_p \leq \tau | R_1 > u \text{ or } R_2 > u \text{ or} \ldots \text{or } R_m > u\}$. This decomposition can be extended to higher risk-aversion levels represented by higher orders of q. This is formalized in the following theorem.

THEOREM 1: *LPM_q for $q \geq 0$ is a combination of different conditional LPM measures on the portfolio. The weight of each of these conditional downside-risk measures is given by a function of the amount of comovements between the assets in the portfolio. More formally,*

$$LPM_q(\tau) = \alpha(u)\gamma_{q,u}(\tau)LPM_{q,u}(\tau) + (1-\alpha(u))\gamma_{q,nu}(\tau)LPM_{q,nu}(\tau), \tag{8}$$

with $\gamma_{q,u}(\tau) = \frac{E[(\tau - R_p)^q | R_p \leq \tau]}{E[(\tau - R_p)^q | R_p \leq \tau, R_1 \leq u, R_2 \leq u, \ldots, R_m \leq u]}$ and
$\gamma_{q,nu}(\tau) = \frac{E[(\tau - R_p)^q | R_p \leq \tau]}{E[(\tau - R_p)^q | R_p \leq \tau, R_1 > u \text{ or } R_2 > u \text{ or} \ldots \text{or } R_m > u]}$.

These different decompositions allow us to disentangle the risk exposure of the portfolio due to comovements produced by the joint collapse of the assets in the portfolio, $\alpha(u)$, from the risk exposure produced by the allocation of weights once the portfolio is under comovements and noncomovements episodes.

The efficient portfolio frontier in models in which risk is measured by probability weighted dispersions below a target is defined by those portfolios minimizing LPM_q measures under the constraints introduced in (2). Bawa and Lindenberg (1977) and Harlow and Rao (1989) show that these measures are consistent with the maximization of preferences of downside-risk averse investors. The general model of this type can be expressed by

Portfolio A dominates Portfolio B in the mean-risk model if and only if $\mu(A) \geq \mu(B)$ and $LPM_q(A) \leq LPM_q(B)$ for $q \geq 0$, with at least one strict inequality.

The proof of this result is given by observing that

$$E[U(R_i; q, \tau), F] = \mu(i) - k\, LPM_q^i(\tau), \tag{9}$$

with $i = A$, B, and k a scale parameter.

Fishburn (1977) shows the existing close connection between the efficiency of LPM_q portfolios and their stochastic dominance over the rest of possible risky portfolios. Before elaborating on this result we will define stochastic dominance between portfolios:

- A first stochastic dominates (FSD) B if $F^A \neq F^B$ and $LPM_0^A(\tau) \leq LPM_0^B(\tau)$ for all $\tau \in \Re$,
- A second stochastic dominates (SSD) B if $F^A \neq F^B$ and $LPM_1^A(\tau) \leq LPM_1^B(\tau)$ for all $\tau \in \Re$,

- A third stochastic dominates (TSD) B if $F^A \neq F^B$, $\mu(A) \geq \mu(B)$, and $LPM_2^A(\tau) \leq LPM_2^B(\tau)$ for all $\tau \in \Re$,

with F^A and F^B the distribution functions of two portfolios A and B.

In particular lemma 1 and theorem 3 in Fishburn (1977) show that if A FSD B then $\mu(A) > \mu(B)$ and $E[v(\cdot), F^A] \geq E[v(\cdot), F^B]$, for every nondecreasing real valued function v; and therefore A dominates B in the mean-risk model for LPM_q measures for all $q \geq 0$. In the same way if A SSD B then $\mu(A) \geq \mu(B)$ and $E[v(\cdot), F^A] \geq E[v(\cdot), F^B]$, for every nondecreasing and concave real valued function v; and therefore A dominates B in the mean-risk model for LPM_q measures for all $q \geq 1$, except when $\mu(A) = \mu(B)$ and $LPM_q^A(\tau) = LPM_q^B(\tau)$ for all τ. Finally, if A TSD B then $\mu(A) \geq \mu(B)$ and $E[v(\cdot), F^A] \geq E[v(\cdot), F^B]$, for every nondecreasing and concave real valued function v for which $-\delta v/\delta x$ is concave, $x \in \mathbb{R}$; and therefore A dominates B in the mean-risk model for LPM_q measures for all $q \geq 2$, except when $\mu(A) = \mu(B)$ and $LPM_q^A(\tau) = LPM_q^B(\tau)$ for all τ.

Therefore these results show that the efficient portfolio sets corresponding to investors minimizing LPM_q measures are a subset of the FSD efficient set for $q \geq 0$; of the SSD efficient set for $q \geq 1$ and of the TSD efficient set for $q \geq 2$; except in the noted cases.

The decomposition in theorem 1 and in particular the definition of $LPM_{q,u}$ measures conditional on the presence of comovements allow us to extend the results in Fishburn to a conditional setting. We need first to define the concept of conditional stochastic dominance:

Definition 1:

- *A first conditional stochastic dominates (FCSD) B if $F_u^A \neq F_u^B$ and $LPM_{0,u}^A(\tau) \leq LPM_{0,u}^B(\tau)$ for all $\tau \leq u$,*
- *A second conditional stochastic dominates (SCSD) B if $F_u^A \neq F_u^B$ and $LPM_{1,u}^A(\tau) \leq LPM_{1,u}^B(\tau)$ for all $\tau \leq u$,*
- *A third conditional stochastic dominates (TCSD) B if $F_u^A \neq F_u^B$, $\mu_u(A) \geq \mu_u(B)$, and $LPM_{2,u}^A(\tau) \leq LPM_{2,u}^B(\tau)$ for all $\tau \leq u$,*

with F_u^A and F_u^B the relevant conditional distribution functions introduced before, and $\mu_u(A) := E[R_p^A | R_1 \leq u, \ldots, R_m \leq u]$ and $\mu_u(B) := E[R_p^B | R_1 \leq u, \ldots, R_m \leq u]$ the corresponding conditional expected values.

Using lemma 1 in Fishburn (1977) we obtain that if A FCSD B then $\mu_u(A) > \mu_u(B)$ and $E[v(\cdot), F_u^A] \geq E[v(\cdot), F_u^B]$, for every nondecreasing real valued function v; if A SCSD B then $\mu_u(A) \geq \mu_u(B)$ and $E[v(\cdot), F_u^A] \geq E[v(\cdot), F_u^B]$, for every nondecreasing and concave real valued function v; and finally, if A TCSD B then $\mu_u(A) \geq \mu_u(B)$ and $E[v(\cdot), F_u^A] \geq E[v(\cdot), F_u^B]$, for every nondecreasing and concave real valued function v for which $-\delta v/\delta x$ is concave, with $x \in \mathbb{R}$.

PROPOSITION 1:

- *If A FCSD B then A dominates B in the mean-risk model defined by $LPM_{q,u}$ measures for all $q \geq 0$.*
- *If A SCSD B then A dominates B in the mean-risk model defined by $LPM_{q,u}$ measures for all $q \geq 1$, except when $\mu_u(A) = \mu_u(B)$ and $LPM_{q,u}^A(\tau) = LPM_{q,u}^B(\tau)$ for all $\tau \leq u$.*
- *If A TCSD B then A dominates B in the mean-risk model defined by $LPM_{q,u}$ measures for all $q \geq 2$, except when $\mu_u(A) = \mu_u(B)$ and $LPM_{q,u}^A(\tau) = LPM_{q,u}^B(\tau)$ for all $\tau \leq u$.*

This result and the decomposition in theorem 1 entails different optimal portfolio choices contingent on the state of the market. An important implication of this proposition is that by testing for different types of conditional stochastic dominance we can sort portfolios in terms of individuals' downside-risk preferences and risk-aversion levels in crises episodes. In order to make the conditions stated before statistically testable we will develop in the next section different hypothesis tests for stochastic and conditional stochastic dominance of different orders.

3 Estimation and Inference

Suppose we have n independent and identically distributed vectors of observations obtained from m different random variables $R_1,\ldots,R_m$. Then, it follows from the definitions in (3) and (4) that natural estimators of $LPM_0(\tau)$ and $LPM_{0,u}(\tau)$, for τ nonstochastic are

$$\widehat{LPM}_0(\tau) := \frac{1}{n}\sum_{i=1}^{n} I(R_{p,i} \leq \tau), \tag{10}$$

and

$$\widehat{LPM}_{0,u}(\tau) := \frac{1}{n_u}\sum_{i=1}^{n} I(R_{p,i} \leq \tau | R_{1,i} \leq u, R_{2,i} \leq u, \ldots, R_{m,i} \leq u), \tag{11}$$

with n_u the number of vectors satisfying $R_1 \leq u, R_2 \leq u, \ldots, R_m \leq u$. The multivariate version of these empirical estimators is employed to estimate $\alpha(u)$. Thus,

$$\widehat{\alpha}(u) := \frac{1}{n}\sum_{i=1}^{n} I(R_{1,i} \leq u, R_{2,i} \leq u, \ldots, R_{m,i} \leq u). \tag{12}$$

The different expected values necessary to compute LPM measures of higher orders are estimated by their corresponding empirical counterparts

$$E[(\tau - \widehat{R_p)^q}|R_p \leq \tau] := \frac{1}{n_p}\sum_{i=1}^{n}(\tau - R_{p,i})^q I(R_{p,i} \leq \tau), \tag{13}$$

and

$$E[(\tau - R_p)^q|R_p \leq \tau, \widehat{R_1 \leq u}, R_2 \leq u, \ldots, R_m \leq u] :=$$

$$= \frac{1}{n'_p}\sum_{i=1}^{n}(\tau - R_{p,i})^q I(R_{p,i} \leq \tau, R_{1,i} \leq u, \ldots, R_{m,i} \leq u), \tag{14}$$

with n_p the number of observations in the sample satisfying $R_p \leq \tau$ and n'_p the number of observations satisfying $R_p \leq \tau$ and $R_1 \leq u, R_2 \leq u, \ldots, R_m \leq u$.

By the strong law of large numbers in the univariate and multivariate setting and by Slutsky's theorem these estimators and linear functions of them are strongly consistent estimators of the population parameters so long as $n'_p \to \infty$, ($n'_p \leq n_p$, $n'_p \leq n_u$). It follows, then, the consistency of the corresponding nonparametric estimators of $LPM_q(\tau)$ and $LPM_{q,u}(\tau)$. It is well known, however, the failure of these nonparametric estimators for small sample sizes. This phenomenon is accentuated in an environment of downside-risk measures defined by very negative threshold values. To solve this, alternative estimators based on extreme value theory, and hence free of model risk, can be obtained by using the Generalized Pareto distribution approximation of the conditional distribution in the tails, as shown in Pickands (1975) and Balkema-de Haan (1974). In particular we derive the form of LPM_q measures and of their consistent semi-parametric estimators for $q = 0, 1, 2$; risk measures of interest for testing first, second and third stochastic dominance.

PROPOSITION 2:

- *For $F(R_p)$ decaying exponentially,*

$$LPM_0(\tau) = \exp^{\left(\frac{\tau}{\sigma_{p,\tau}}\right)}[1 + o(1)],$$

$$LPM_1(\tau) = \sigma_{p,\tau} exp^{\left(\frac{\tau}{\sigma_{p,\tau}}\right)}\left[1 + o\left(1\right)\right], \tag{15}$$

and

$$LPM_2(\tau) = 2\sigma_{p,\tau}^2 exp^{\left(\frac{\tau}{\sigma_{p,\tau}}\right)}\left[1 + o\left(1\right)\right], \tag{16}$$

with $\sigma_{p,\tau}^2 := V(R_p|R_p \leq \tau)$, and $V(\cdot)$ denoting the variance operator.

- *For $F(R_p)$ polynomially decaying,*

$$LPM_0(\tau) = (-\tau)^{-1/\xi_p}[1 + o(1)],$$

$$LPM_1(\tau) = \frac{\sigma_{p,\tau}}{1 - \xi_P} (-\tau)^{-1/\xi_p} [1 + o(1)], \tag{17}$$

and

$$LPM_2(\tau) = \frac{2\sigma^2_{p,\tau}}{(1 - \xi_p)(1 - 2\xi_p)} (-\tau)^{-1/\xi_p} [1 + o(1)], \tag{18}$$

with ξ_p denoting the tail index of portfolio P.

The estimates of these measures are calculated very easily by computing the empirical counterparts of the unknown parameters of interest $\sigma^2_{p,\tau}$ and ξ_P. Estimation of these parameters is standard in the statistical literature, see for example Embrechts, Klüppelberg and Mikosch (1997) for a review of estimators for the tail index.

3.1 A Hypothesis Test for Stochastic and Conditional Stochastic Dominance

In what follows we will concentrate on testing stochastic dominance and conditional stochastic dominance of orders zero, one and two. This is an open problem widely investigated in the statistical and financial literature, see McFadden (1989), Larsen and Resnick (1993) Kaur, Rao and Singh (1994), Anderson (1996), Davidson and Duclos (2000), Barret and Donald (2003) or recently Davidson and Duclos (2006). Our approach differs from these influential papers in two basic aspects: in contrast to most of these papers we present a testing framework that allows to test for first, second and third stochastic dominance using simple modifications of the test statistic and the asymptotic theory, and that makes allowance for dependence between portfolios; and second, our emphasis is on testing conditional stochastic dominance defined by the presence of comovements between the assets in the portfolio.

Our test statistic is of a Kolmogorov-Smirnov type and shares the spirit of the test statistic proposed in Barret and Donald (2003). Therefore it differs from most of the other existing methods that are based on the minimum distance rather than on the maximum, see for example Kaur, Rao and Singh (1994). This is possible in our environment due to the choice of a utility function (1) that is increasing for $q = 0$ and nondecreasing and concave for $q > 0$. Thereby the results in Fishburn (1977) allow us to focus on the hypothesis test

$$H_{0,\gamma} : LPM^A_\gamma(\tau) \leq LPM^B_\gamma(\tau), \quad \text{for all } \tau \in \mathbb{R},$$

rather than on the strict inequality, for testing first ($\gamma = 0$), second ($\gamma = 1$) and third ($\gamma = 2$)

stochastic dominance between two portfolios A and B. Note that as shown before, under $H_{0,0}$ A dominates B in the mean-risk sense for risk-neutral and risk-averse investors, under $H_{0,1}$ A dominates B for risk-averse investors except when $\mu(A) = \mu(B)$, and under $H_{0,2}$ and $\mu(A) \geq \mu(B)$ A dominates B for risk-averse investors with increasing absolute risk aversion level.

PROPOSITION 3: *Suppose we have n independent and identically distributed observations from a random variable R, and let $\widehat{LPM}_{\gamma}(\tau)$ be any $\sqrt{n}-$consistent estimator of $LPM_{\gamma}(\tau)$. Then*

$$\sqrt{n}\left(\widehat{LPM}_{\gamma}(\tau) - LPM_{\gamma}(\tau)\right) \xrightarrow{d} N(0, E[(\tau - R)^{\gamma}|R \leq \tau]^2 F(\tau)(1 - F(\tau))), \tag{19}$$

for all fixed τ in the real line.

To derive the asymptotic theory relevant to the hypothesis test we need the following assumptions.

Assumption A.1: Let A and B denote two portfolios with returns characterized by two random variables R_p^A and R_p^B respectively. Denote $F^{A,B}(\tau,\tau) := P\{R_p^A \leq \tau, R_p^B \leq \tau\}$, $k_{\gamma}^{i}(\tau) = E[(\tau - R_p^i)^{\gamma}|R_p^i \leq \tau]$ for $i = A, B$, and $k_{\gamma}^{A,B}(\tau,\tau) := E[(\tau - R_p^A)^{\gamma}(\tau - R_p^B)^{\gamma}|R_p^A \leq \tau, R_p^B \leq \tau]$. Then

$$E[(\tau - R_p^A)^{\gamma}|R_p^A \leq \tau]E[(\tau - R_p^B)^{\gamma}|R_p^B \leq \tau]F^A(\tau)(1 - F^A(\tau))F^B(\tau)(1 - F^B(\tau)) > \left(k_{\gamma}^{A,B}(\tau,\tau)F^{A,B}(\tau,\tau) - k_{\gamma}^{A}k_{\gamma}^{B}F^A(\tau)F^B(\tau)\right)^2,$$

for all $\tau \in \mathbb{R}$.

Assumption A.2: F^A and F^B are continuous distribution functions.

Assumption A.1 implies that result (19) can be extended to describe the bivariate asymptotic distribution of $\widehat{LPM}_{\gamma}^{A}(\tau)$ and $\widehat{LPM}_{\gamma}^{B}(\tau)$. Assumption A.2 implies the uniqueness of the copula function linking the joint distribution $F^{A,B}$ with its marginal distribution funtions F^A and F^B, see Sklar (1959) and Nelsen (1998) for more details and examples. In particular, A.2 implies the following representation $F^{A,B}(\tau,\tau) := C(F^A(\tau), F^B(\tau))$ with C denoting a copula function describing the dependence structure on the $[0,1]^2$ domain. On the other hand, if F^A or F^B show some discontinuity in their domain the copula representation is no longer unique.

Proposition 3 can be extended under A.1 and A.2 to two variables yielding the following result

$$\sqrt{n}\left(\widehat{LPM}_{\gamma}^{A}(\tau) - LPM_{\gamma}^{A}(\tau), \widehat{LPM}_{\gamma}^{B}(\tau) - LPM_{\gamma}^{B}(\tau)\right) \xrightarrow{d}$$

$$N\left(\left(\begin{array}{c}0\\0\end{array}\right),\left(\begin{array}{cc}(k_\gamma^A(\tau))^2F^A(\tau)(1-F^A(\tau)) & k_\gamma^{A,B}(\tau,\tau)C(F^A,F^B)-k_\gamma^Ak_\gamma^BF^A(\tau)F^B(\tau)\\ k_\gamma^{A,B}(\tau,\tau)C(F^A,F^B)-k_\gamma^Ak_\gamma^BF^A(\tau)F^B(\tau) & (k_\gamma^B(\tau))^2F^B(\tau)(1-F^B(\tau))\end{array}\right)\right), \quad (20)$$

for all fixed $\tau \in \mathbb{R}$, and $C(F^A, F^B)$ denoting $C(F^A(\tau), F^B(\tau))$. Note that by the Cramer-Wold device it is immediate to observe that the limit distribution of the difference between the random variables also converges to a normal distribution. Thus,

$$\sqrt{n}\left((\widehat{LPMdiff}_\gamma(\tau) - LPMdiff_\gamma(\tau)\right) \xrightarrow{d} N\left(0, V(\tau)\right) \quad (21)$$

with $LPMdiff_\gamma(\tau) := LPM_\gamma^A(\tau) - LPM_\gamma^B(\tau)$ and where $V(\tau) = (k_\gamma^A(\tau))^2F^A(\tau)(1-F^A(\tau)) + (k_\gamma^B(\tau))^2F^B(\tau)(1-F^B(\tau)) - 2(k_\gamma^{A,B}(\tau,\tau)C(F^A(\tau),F^B(\tau)) - k_\gamma^A(\tau)k_\gamma^B(\tau)F^A(\tau)F^B(\tau))$.

Under the hypothesis $F^A(\tau) = F^B(\tau)$ for a fixed τ and for $\gamma = 0$ the limit distribution (21) is well defined if and only if $F^B(\tau) > C(F^B(\tau), F^B(\tau))$. Since we are interested in testing the hypothesis for every τ in the real line rather than for a nonstochastic fixed τ this condition will be necessary but not sufficient. We need to extend (21) to the associated continuous random process indexed by $\tau \in \mathbb{R}$. Proposition 4 provides the extension to a multivariate setting defined by a vector of τ_i values, and Proposition 5 shows the tightness of the process. Corollary 1 derives the asymptotic distribution of the supremum functional.

PROPOSITION 4: *Suppose we have a partition of the real line given by $-\infty < \tau_1 < \tau_2 < \ldots < \tau_t < \infty$, and n independent and identically distributed observations from two random variables R_p^A and R_p^B. Let $\widehat{LPMdiff}_\gamma(\tau)$ be the consistent estimator of $LPMdiff_\gamma(\tau)$ introduced above. Then, under A.1 and A.2,*

$$\sqrt{n}\left(\widehat{LPMdiff}_\gamma(\tau_1) - LPMdiff_\gamma(\tau_1), \ldots, \widehat{LPMdiff}_\gamma(\tau_t) - LPMdiff_\gamma(\tau_t)\right) \xrightarrow{d} \left(G(\tau_1), \ldots, G(\tau_t)\right), \quad (22)$$

with the vector on the right following a multivariate normal distribution with mean zero and covariance matrix given by

$$E[G(\tau_i)G(\tau_j)] =$$
$$k_\gamma^A(\tau_i)k_\gamma^A(\tau_j)\left(F^A(\tau_i \wedge \tau_j) - F^A(\tau_i)F^A(\tau_j)\right) + k_\gamma^B(\tau_i)k_\gamma^B(\tau_j)\left(F^B(\tau_i \wedge \tau_j) - F^B(\tau_i)F^B(\tau_j)\right) -$$
$$\left(k_\gamma^{A,B}(\tau_i,\tau_j)C(F^A(\tau_i),F^B(\tau_j)) - k_\gamma^A(\tau_i)k_\gamma^B(\tau_j)F^A(\tau_i)F^B(\tau_j)\right) -$$
$$\left(k_\gamma^{A,B}(\tau_j,\tau_i)C(F^A(\tau_j),F^B(\tau_i)) - k_\gamma^A(\tau_j)k_\gamma^B(\tau_i)F^A(\tau_j)F^B(\tau_i)\right),$$

for all $\tau_i, \tau_j \in \mathbb{R}$.

PROPOSITION 5: *Under A.1 and A.2, the sequence of empirical processes in (22) converges in* *distribution in the Skorohod space $D[-\infty, \infty]$, equipped with the uniform norm, to a Gaussian process $G(\tau)$ with zero mean and covariance functions as in the preceding display.*

Now, by the continuous mapping theorem we obtain the following result.

COROLLARY 1:

$$\sqrt{n} \sup_{\tau \in \mathbb{R}} (\widehat{LPMdiff}_{\gamma}(\tau) - LPMdiff_{\gamma}(\tau)) \xrightarrow{d} \sup_{\tau \in \mathbb{R}} G(\tau), \tag{23}$$

with $G(\tau)$ a Gaussian process with zero mean and covariance functions as shown above.

We are ready to introduce our test statistic for the hypothesis of stochastic dominance. Note that the null hypothesis in $H_{0,\gamma}$ implies that A dominates B, and the alternative hypothesis rejection of stochastic dominance of B by A. Other papers proposing this type of null hypothesis are McFadden (1989), Anderson (1996) or Davidson and Duclos (2000) amongst others.

THEOREM 2: *Let $T_n(\gamma) := \sqrt{n} \sup_{\tau \in \mathbb{R}} \widehat{LPMdiff}_{\gamma}(\tau)$ be a family of test statistics indexed by γ for $H_{0,\gamma}$ for $\gamma = 0, 1, 2$. Then, its asymptotic distribution under $H_{0,\gamma}$ is the supremum of a Gaussian process with zero mean and covariance function given by*

$$E[G(\tau_i)G(\tau_j)] =$$
$$2k_{\gamma}^{B}(\tau_i)k_{\gamma}^{B}(\tau_j)F^{B}(\tau_i \wedge \tau_j) - k_{\gamma}^{A,B}(\tau_i, \tau_j)C(F^{B}(\tau_i), F^{B}(\tau_j)) - k_{\gamma}^{A,B}(\tau_j, \tau_i)C(F^{B}(\tau_j), F^{B}(\tau_i)),$$

with τ_i, $\tau_j \in \mathbb{R}$, for all i, j. Under independence between A and B the covariance function boils down to

$$E[G(\tau_i)G(\tau_j)] = 2\left\{k_{\gamma}^{B}(\tau_i)k_{\gamma}^{B}(\tau_j)\left(F^{B}(\tau_i \wedge \tau_j) - F^{B}(\tau_i)F^{B}(\tau_j)\right)\right\}. \tag{24}$$

Under A.1 and A.2, the limit distribution of this process is well defined if and only if $F^{B}(\tau) > C(F^{B}(\tau), F^{B}(\tau))$ for every $\tau \in \mathbb{R}$. The rejection region of this test lies on the right tail of the asymptotic distribution that is determined for an α size by a cut-off point $x_{1-\alpha}$ defined as

$$x_{1-\alpha} := \sup_{x \in \mathbb{R}} \{x \,|\, \lim_{n \to \infty} P\{T_n(\gamma) > x\} \geq \alpha\}, \tag{25}$$

where $x \in \mathbb{R}$. The corresponding asymptotic power for a significance level α is given by

$$p = \lim_{n \to \infty} P\{\sqrt{n} \sup_{\tau \in \mathbb{R}} \left(\widehat{LPMdiff}_{\gamma}(\tau) - LPMdiff_{\gamma}(\tau)\right) > x_{1-\alpha}\}. \tag{26}$$

For first stochastic dominance the limit process in theorem 2 is a function of a F^B-Brownian bridge, that therefore has a distribution that is known and tabulated. For higher orders of stochastic dominance one has to rely on simulation techniques to approximate the asymptotic critical values. Note that the consistency of the nonparametric estimates of $F^B(\cdot)$ and $C(\cdot,\cdot)$ results in consistent estimates of $x_{1-\alpha}$ when these parameters are not known. Bootstrap resampling techniques offer, in principle, a good alternative to approximate in finite samples the size of the test. For the study of the power curve, however, this technique fails to approximate the null distribution since one does not know whether $LPM_q^A \leq LPM_q^B$ or $LPM_q^B \leq LPM_q^A$ and thereby whether the bootstrap approximates the null or the alternative distribution.

These results can be easily extended to testing conditional stochastic dominance and with it the mean-risk dominance set in comovements episodes. For ease of exposition we will assume that both portfolios have the same number of vectors of observations n_u below the threshold u and are comprised by the same assets. This assumption on the number of vectors in n below u is not very restrictive since one can always define the threshold in terms of the number of observations n_u in the multivariate tail.

COROLLARY 2: *Let u be a fixed threshold value in the real line. Then*

$$\sqrt{n_u} \sup_{\tau \in (-\infty,u]} (\widehat{LPMdiff}_{\gamma,u}(\tau) - LPMdiff_{\gamma,u}(\tau)) \xrightarrow{d} \sup_{\tau \in (-\infty,u]} G_u(\tau), \tag{27}$$

with $LPMdiff_{\gamma,u}(\tau) := LPM_{\gamma,u}^A(\tau) - LPM_{\gamma,u}^B(\tau)$, and where $G_u(\tau)$ is a Gaussian process with zero mean and covariance functions given by

$$\begin{aligned} E[G_u(\tau_i)G_u(\tau_j)] = k_{\gamma,u}^A(\tau_i)k_{\gamma,u}^A(\tau_j)\left(F_u^A(\tau_i \wedge \tau_j) - F_u^A(\tau_i)F_u^A(\tau_j)\right) + \\ k_{\gamma,u}^B(\tau_i)k_{\gamma,u}^B(\tau_j)\left(F_u^B(\tau_i \wedge \tau_j) - F_u^B(\tau_i)F_u^B(\tau_j)\right) - \\ \left(k_{\gamma,u}^{A,B}(\tau_i,\tau_j)C_u(F_u^A(\tau_i),F_u^B(\tau_j)) - k_{\gamma,u}^A(\tau_i)k_{\gamma,u}^B(\tau_j)F_u^A(\tau_i)F_u^B(\tau_j)\right) - \\ \left(k_{\gamma,u}^{A,B}(\tau_j,\tau_i)C_u(F_u^A(\tau_j),F_u^B(\tau_i)) - k_{\gamma,u}^A(\tau_j)k_{\gamma,u}^B(\tau_i)F_u^A(\tau_j)F_u^B(\tau_i)\right). \end{aligned}$$

with $\tau_i,\ \tau_j \leq u$, for all i,j, $F_u^{A,B}(\tau_i,\tau_j) := P\{R_p^A \leq \tau_i, R_p^B \leq \tau_j | R_1 \leq u,\dots,R_m \leq u\}$, $k_{\gamma,u}^i(\tau) := E[(\tau - R_p^i)^\gamma | R_p^i \leq \tau, R_1 \leq u,\dots,R_m \leq u]$ and $k_{\gamma,u}^{A,B}(\tau_i,\tau_j) := E[(\tau_i - R_p^A)^\gamma(\tau_j - R_p^B)^\gamma | R_p^A \leq \tau, R_p^B \leq \tau, R_1 \leq u,\dots,R_m \leq u]$. The copula $C_u(F_u^A(\tau_i),F_u^B(\tau_j)) := F_u^{A,B}(\tau_i,\tau_j)$ describes the dependence structure between A and B conditional on the event $R_1 \leq u,\dots,R_m \leq u$.

The relevant hypothesis test is

$$H_{0,\gamma,u} : LPM_{\gamma,u}^A(\tau) \leq LPM_{\gamma,u}^B(\tau) \text{ for all } \tau \leq u.$$

The following theorem provides the asymptotic theory of the test.

THEOREM 3: *Let* $T_{n_u}(\gamma) := \sqrt{n_u} \sup_{\tau \in (-\infty, u]} \widehat{LPMdiff}_{\gamma,u}(\tau)$ *be a test statistic for* $H_{0,\gamma,u}$ *for* $\gamma = 0, 1, 2$ *and* u *a threshold value. Then, its asymptotic distribution is the supremum of the Gaussian process* G_u *with zero mean and covariance function given by*

$$E[G_u(\tau_i)G_u(\tau_j)] = 2\left\{k^B_{\gamma,u}(\tau_i)k^B_{\gamma,u}(\tau_j)F^B_u(\tau_i \wedge \tau_j) - k^{A,B}_{\gamma,u}(\tau_i, \tau_j)C_u(F^B_u(\tau_i), F^B_u(\tau_j))\right\},$$

with τ_i, $\tau_j \in \mathbb{R}$, *for all* i, j. *Under tail-independence between* A *and* B *the covariance function boils down to*

$$E[G_u(\tau_i)G_u(\tau_j)] = 2\left\{k^B_{\gamma,u}(\tau_i)k^B_{\gamma,u}(\tau_j)\left(F^B_u(\tau_i \wedge \tau_j) - F^B_u(\tau_i)F^B_u(\tau_j)\right)\right\}. \quad (28)$$

The size and power of the different tests are obtained from the appropriate modifications of the formulas derived in the unconditional setting. These results are illustrated in the Monte Carlo simulation section.

These tests extend in two ways the existing methods for testing stochastic dominance found in the literature. First, by deriving a testing framework not only for FSD but also for SSD, TSD, and higher orders of dominance and that makes allowance for dependence between portfolios; and second by introducing and deriving a theory for testing conditional stochastic dominance that allows to discriminate between portfolios in crisis episodes of the market. The implications of these techniques in optimal portfolio theory, as we have seen in preceding sections, are of much interest. A simple application with data from US and UK financial markets is described in Section 5. Next section illustrates via simulation experiments the findings of this section.

4 Mote-Carlo Simulation Experiments

In this section we consider a small Monte Carlo experiment to gauge the extent to which the preceding asymptotic arguments hold in small samples. We carry out five different experiments in an effort to illustrate the theory introduced before and also to replicate real diversification strategies among investment portfolios.

The first experiment is aimed at illustrating the power of the test statistic $T_n(0)$ for a family of independent Gaussian distributions with different variance parameters, and of independent Student-t distributions where the alternative hypotheses are defined by the degrees of freedom. Figure 4.1 shows two interesting facts: a) the consistency of this test, since the power of the test increases with the sample size, and b) stochastic dominance in an elliptical environment is completely determined by the variance of the distribution (under Gaussianity) and by the degrees of freedom (Student-t) if the variance remains constant across distribution functions.

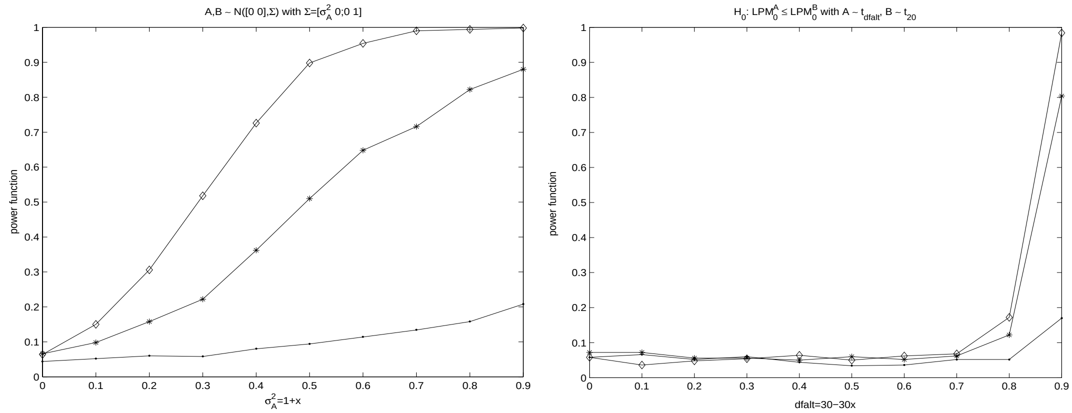

Figure 4.1. *Power functions for $H_{0,0}$. Left panel entertains a family of alternative hypotheses described by normal distributions with variance σ_A^2. Right panel entertains a family of alternatives given by Student-t with $dfalt$ degrees of freedom. $n = 100$ is plotted by $(.-)$, $n = 500$ with $(*-)$ and $n = 1000$ with $(o-)$. The asymptotic critical value at 5% is generated from a partition of 100 points of the real line and B=1000 iterations. Monte Carlo simulations $mc = 500$.*

A second battery of simulations shows the size and power of this test for the case of dependence between A and B. We consider the Gumbel copula $C(v_1, v_2) = \frac{v_1 v_2}{v_1 + v_2 - v_1 v_2}$ with $v_1 = F^A$ and $v_2 = F^B$ two $[0, 1]-$uniform random variables. The asymptotic distribution of the test is simulated via the supremum of the process introduced in theorem 2. In order to calculate the power function we generate random samples from a bivariate Gumbel copula with Gaussian margins, see Nelsen (1998) for details on generating random variates, and count the number of times the test statistic exceeds the simulated critical value at 5%. The alternative hypotheses are given by a marginal Gaussian distribution for A with higher variance than the marginal corresponding to B.

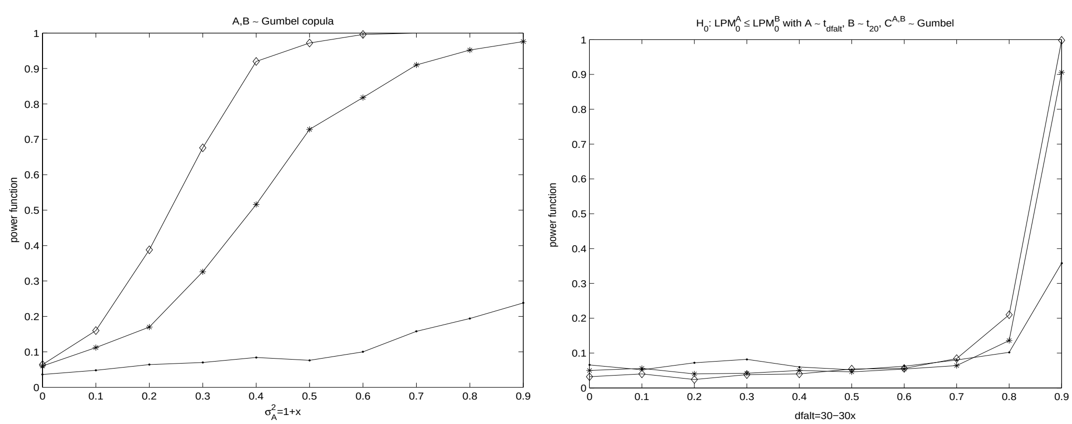

Figure 4.2. *Power functions for $H_{0,0}$ when the joint distribution of A and B is Gumbel with Gaussian margins. Left panel entertains a family of alternative hypotheses described by normal distributions with variance σ_A^2. Right panel entertains a family of alternatives given by Student-t with $dfalt$ degrees of freedom. $n = 100$ is plotted by $(.-)$, $n = 500$ with $(*-)$ and $n = 1000$ with $(o-)$. The asymptotic critical value at 5% is generated from a partition of 100 points of the real line and B=1000 iterations. Monte Carlo simulations $mc = 500$.*

The third experiment covers the tests for second order stochastic dominance. The conclusions from figure 4.3 are along the lines of those for first stochastic dominance. We observe in this case, however, an slightly overestimated size. After considering different values of the number of Monte Carlo iterations, partitions of the real line for computing the asymptotic critical value at 5%, and simulating different sample sizes we have observed that this distortion in the estimated power function is largely due to the effect of not considering a sufficiently large partition of the real line for simulating the asymptotic distribution and computing the appropriate critical value. Thereby the rationale to partition the real line in 1000 points rather than in 100 as in the previous examples. Due to computational issues, however, we do not present a further thinner partition that would provide a more accurate approximation of the asymptotic critical value. It is also interesting to observe the lack of power of this test for alternative hypothesis defined by Student-t distributions with heavier tails than that with $df = 20$.

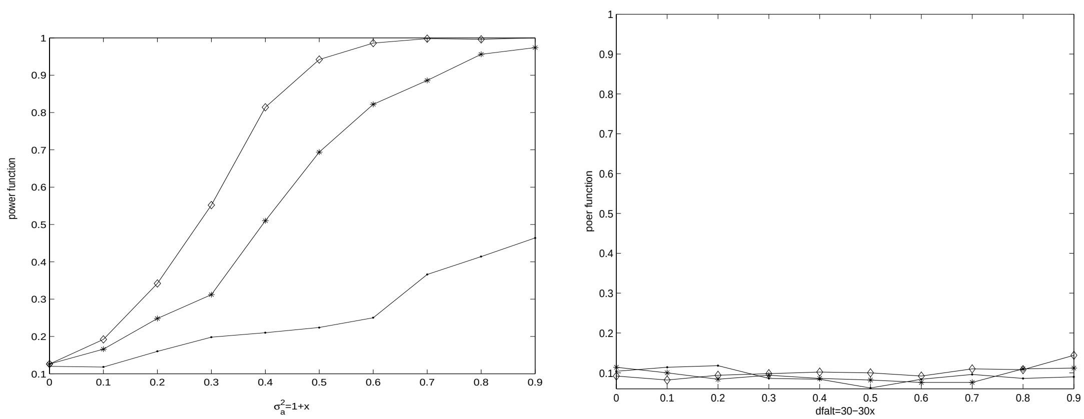

Figure 4.3. *Power functions for $H_{0,1}$. Left panel entertains a family of alternative hypotheses described by normal distributions with variance σ_A^2. Right panel entertains a family of alternatives given by Student-t with $dfalt$ degrees of freedom. $n = 100$ is plotted by $(.-)$, $n = 500$ with $(*-)$ and $n = 1000$ with $(o-)$. The asymptotic critical value at 5% is generated from a partition of 100 points of the real line and B=1000 iterations. Monte Carlo simulations $mc = 500$.*

The following simulation exercise aims at studying the tests for first and second stochastic dominance conditional on comovements. In contrast to the unconditional case the asymptotic critical value depends on nuisance parameters $F_u^B(\cdot)$ and $C_u(\cdot,\cdot)$ that are approximated by using the nonparametric estimates discussed in Section 3. The conclusions from these plots are along the lines of the unconditional exercise.

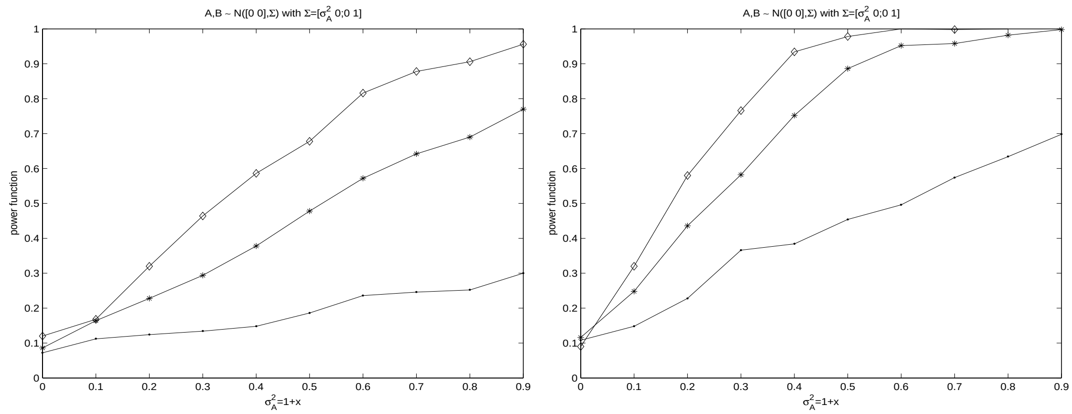

Figure 4.4. *Power functions for $H_{0,0,0}$ on the left panel and $H_{0,1,0}$ on the right panel. Left panel entertains a family of alternative hypotheses described by normal distributions with variance σ_A^2. Right panel entertains a family of alternatives given by Student-t with $dfalt$ degrees of freedom. $n = 100$ is plotted by $(.-)$, $n = 500$ with $(*-)$ and $n = 1000$ with $(o-)$. The asymptotic critical value at 5% is generated from a partition of 100 points of the real line and B=1000 iterations. Monte Carlo simulations $mc = 500$.*

Finally figure 4.5 shows a first stochastic dominance efficiency analysis simulation exercise between two portfolios A and B consisting of the same two assets, independent and normally distributed with variances a and b. These two portfolios A and B differ in the weight on each asset. The null hypothesis in both plots is the equally weighted portfolio with variance $0.25(a+b)$, and the alternatives consist of every other combination of weights satisfying $w_{A,1} + w_{A,2} = 1$, and $0 \leq w_{A,1}, w_{A,2} \leq 1$, with variance $aw_1^2 + bw_2^2$. From the power function one can derive the set of portfolios that are first order stochastic dominated by the equally weighted portfolio.

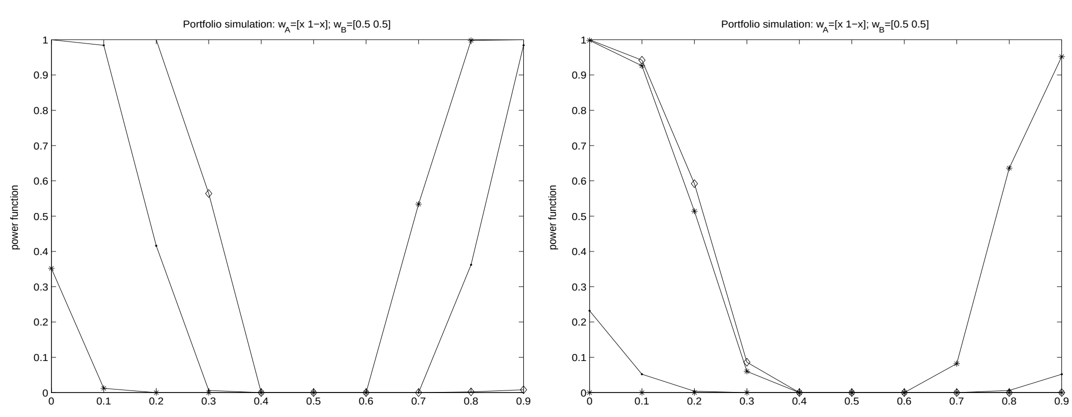

Figure 4.5. *Simulation of Portfolio $A := w_{A,1}R_1 + w_{A,2}R_2$ and Portfolio $B := 0.5R_1 + 0.5R_2$. (R_1, R_2) follows a bivariate normal distribution with covariance matrix $\Sigma = [a\,0; 0\,b]$. $H_{0,0}$ =Portfolio B (left panel) and $H_{0,0,0}$ =:Portfolio B (right panel). $w_{A,1} \in [0,1]$, $w_{A,2} = 1 - w_{A,1}$. $(.-)$ describes A with $(a,b) = (1,1)$, $(*-)$ describes A with $(a,b) = (2,1)$ and $(o-)$ A with $(a,b) = (1,2)$. $n = 1000$ observations. The asymptotic critical value is generated from a partition of 100 points and B=1000 iterations. Monte Carlo simulations $mc = 500$.*

This last exercise is useful as introduction of the application of this methodology to real data. We will be interested in comparing the risk of two investment portfolios in normal and crises episodes of the market, for this we will use the mean-variance methodology as benchmark for investment decisions.

5 Mean-variance v Mean-risk efficiency under Comovements

This application consists of two examples. The first example studies an investment portfolio comprised by four assets describing equity, debt and currency markets in US and UK. These are Dow-Jones Stock Index (*DJSI*), *FTSE-100* Index, Dow-Jones Corporate bonds Index with 2-years maturity (*DJCB*), and the dollar/sterling pound exchange rate (*US/UK*) to complement potential diversification benefits from downturns/upturns in US/UK economic activity. The data set we propose to use spans the period 22/01/2001 - 24/09/2004 and consists of log-returns, $r_t = 100{*}log\, P_t/P_{t-1}$, with P_t prices obtained from www.freelunch.com. The vector of optimal weights for each investment strategy is reported in table 5.1 and the corresponding estimates of the different risk measures in figure 5.1.

u / risk measure	$LPM_{0,u}^{P}(\tau)$	$LPM_{1,u}^{P}(\tau)$	$\sigma_{P,u}^{2}$
$u = \infty$	[0.05 0.85 0.05 0.05]	[0.10 0.05 0.85 0.00]	[0.10 0.05 0.85 0.00]
$u = 0$	[0.05 0.05 0.05 0.85]	[0.10 0.05 0.85 0.00]	[0.10 0.05 0.85 0.00]
$u = 1$	[0.05 0.05 0.05 0.85]	[0.85 0.05 0.10 0.00]	[0.10 0.05 0.85 0.00]

Table 5.1. *Optimal weights obtained from the different mean-risk efficient sets for* $\mu(P) \leq 2$. *The assets comprising the portfolios are* $(US/UK, FTSE, DJBC, DJSI)$. $LPM_{0,\infty} := LPM_0$, $LPM_{1,\infty} := LPM_1$ *and* $\sigma^2_{P,\infty} := \sigma^2_P$. $\tau = 2$.

The upper row from table 5.1 shows that the mean-variance efficient portfolio is also the efficient portfolio obtained from minimizing $LPM_1(\tau)$ with $\tau = 2$. From the tests for first stochastic dominance we observe that there is statistical evidence to reject the hypothesis for both $H_{0,0} : LPM_0^{w_{mv}}(\tau) \leq LPM_0^{w_0}(\tau)$ and for $H^*_{0,0} : LPM_0^{w_0} \leq LPM_0^{w_{mv}}$ with w_0 and w_{mv} the efficient weights from each investment strategy, and for all $\tau \in \mathbb{R}$. This implies that neither portfolio dominates the other in distribution, and therefore there is no optimal ranking of portfolios across $\tau \in \mathbb{R}$ for risk-neutral investors. This can be also seen from the plots of the unconditional LPM_0 measures in figure 5.1.

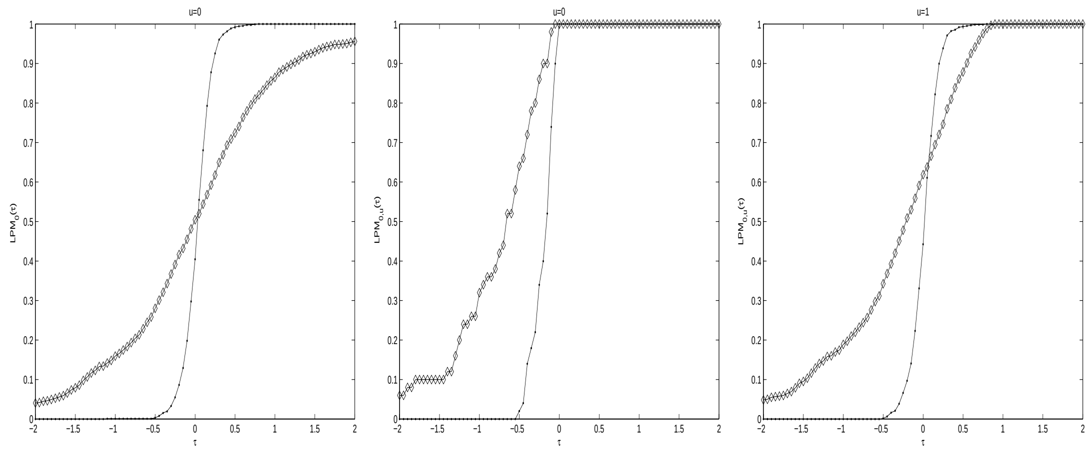

Figure 5.1. *Nonparametric estimates of the conditional risk measures LPM_0 (left panel) and $LPM_{0,u}$ (middle and right panels). The curves corresponding to Portfolio w_0 and $w_{0,u}$ are plotted with $(.-)$, to Portfolio w_1 and $w_{1,u}$ with $(*-)$ and to Portfolio w_{mv} with $(o-)$. The assets comprising the portfolio are $(US/UK, FTSE, DJBC, DJSI)$ and the relevant period 22/1/2001 - 24/09/2004.*

To see if there exists such ranking for risk-averse investors we test for second stochastic dominance, $H_{0,1} : LPM_1^{w_{mv}} \leq LPM_1^{w_0}$. In this case we obtain a critical value of 1.67 from the Monte-Carlo approximation and a test statistic of -2.13, leading us not to reject the null hypothesis. A simple hypothesis of means between portfolios also rejects the equality of expected returns, and we can conclude that the mean-variance efficient set second stochastic dominates the other portfolio, and therefore is preferred by risk-averse investors. In other words, risk-averse investors adopting optimal investment strategies will be mean-variance minimizing agents.

Figure 5.1 also sheds light about first stochastic dominance between portfolios conditional on comovements. The test $H_{0,0,u} : LPM_{0,u}^{w_{mv,u}}(\tau) \leq LPM_{0,u}^{w_{0,u}}(\tau)$ for all $\tau \leq u$, with $w_{mv,u}$ and $w_{0,u}$ the optimal weights conditional on comovements formalizes this result. With the data available, we do not find statistical evidence to reject the null hypothesis of conditional first stochastic dominance, therefore concluding that the mean-variance efficient portfolio first stochastic dominates the other efficient portfolio in distress episodes of the market. This result implies that the mean-variance efficient portfolio is also superior to the downside-risk portfolio in comovements episodes of the market for risk-neutral as well as for risk-averse investors. For a higher threshold $u = 1$ the decomposition on the right panel of figure 5.1 and the corresponding hypothesis tests show results consistent with the unconditional case. To understand this result it is important to remark that while the probability of comovements defined by $u = 0$ is 0.05, the probability defined by $u = 1$ is 0.68.

The second example studies a portfolio of risky and heavily traded stocks in the US economy that cover very different and important sectors: Microsoft (MSFT), General Electric (GE), Bank of America Corporation (BAC) and Verizon Communications (VZ). The data set we propose to use spans the period 02/01/2000 - 30/12/2007 and are obtained from Yahoo Finance website. In contrast to the previous case each asset is not a diversified instrument per se and can be dramatically affected by negative and positive idiosyncratic shocks yielding distribution functions very different from the Gaussian one. This fact can invalidate approximations of the distribution of the corresponding portfolio given by the Gaussian distribution, and can therefore cast some doubt about the supremacy of the mean-variance efficient set. The vector of optimal weights in this case is reported in table 5.2 and the corresponding risk measures in figure 5.2.

u / risk measure	$LPM^P_{0,u}(\tau)$	$LPM^P_{1,u}(\tau)$	$\sigma^2_{P,u}$
$u = \infty$	[0.05 0.85 0.05 0.05]	[0.30 0.20 0.35 0.15]	[0.20 0.15 0.30 0.35]
$u = 0$	[0.05 0.05 0.05 0.85]	[0.05 0.05 0.05 0.85]	[0.05 0.05 0.50 0.40]
$u = 1$	[0.05 0.05 0.05 0.85]	[0.05 0.05 0.05 0.85]	[0.15 0.10 0.35 0.40]

Table 5.2. *Optimal weights obtained from the different mean-risk efficient sets for $\mu_p \leq 2$. The assets comprising the portfolios are $(GE, MSFT, VZ, BAC)$. $LPM_{0,\infty} := LPM_0$, $LPM_{1,\infty} := LPM_1$ and $\sigma^2_{P,\infty} := \sigma^2_P$. $\tau = 2$.*

The left panel in figure 5.2 indicates that there is no possible ranking of portfolios in terms of first order stochastic dominance since the relevant distribution functions cross each other.

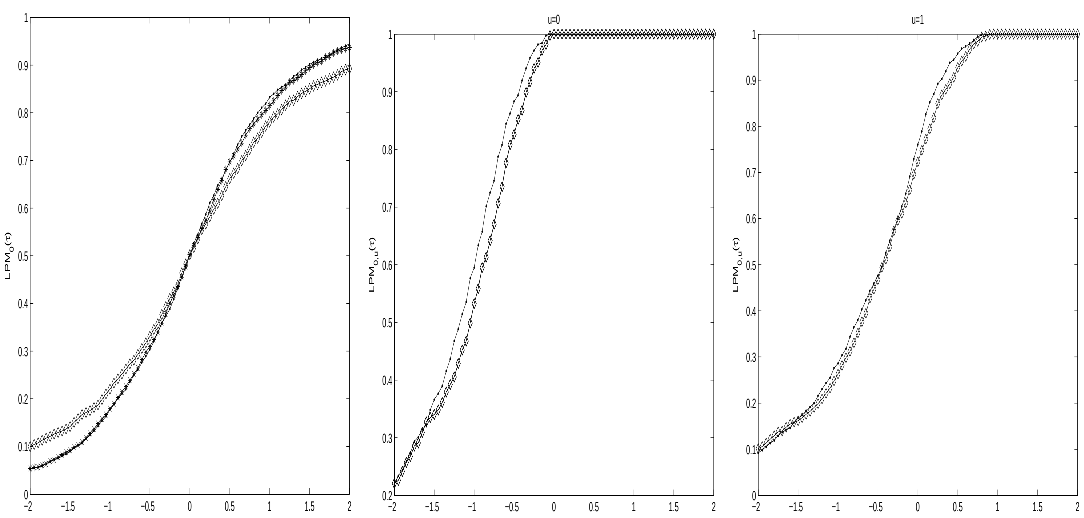

Figure 5.2. *Nonparametric estimates of the conditional risk measures LPM_0 (left panel) and $LPM_{0,u}$ (middle and right panels). The curves corresponding to Portfolio w_0 and $w_{0,u}$ are plotted with $(.-)$, to Portfolio w_1 and $w_{1,u}$ with $(*-)$ and to Portfolio w_{mv} with $(o-)$. The assets comprising the portfolio are $(GE, MSFT, VZ, BAC)$ and the relevant period 02/01/2000 - 30/12/2007.*

This is formalized with the results below of the tests corresponding to the three possible combinations between portfolios.

$H_{0,0}: A \leq B$	LPM_0^P	LPM_1^P	σ_P^2
$A: LPM_0^P$	-	1	1
$A: LPM_1^P$	1	-	0
$A: \sigma_P^2$	1	0	-

$H_{0,1}: A \leq B$	LPM_0^P	LPM_1^P	σ_P^2
$A: LPM_0^P$	-	1	1
$A: LPM_1^P$	0	-	0
$A: \sigma_P^2$	0	0	-

Table 5.3. *Empirical rejection of the tests $H_{0,0}$ and $H_{0,1}$. 0 denotes acceptance of the relevant hypothesis and 1, rejection of the hypothesis at 5% significance level.*

The left table shows that Portfolio w_0 and w_1 are not first stochastic dominated by each other, and that Portfolio w_1 and w_{mv} are not statistically significantly different. The right table also shows that the mean-variance efficient portfolio and w_1 efficient portfolio are very similar. More importantly, both portfolios second stochastic dominate Portfolio w_0. Therefore risk-averse investors prefer these strategies to the portfolio derived from minimizing a particular risk measure $LPM_0(\tau)$ for $\tau = 2$.

The results under comovements are diametrically different. The portfolios derived from downside-risk measures, characterized now by $w_{0,0}$ and $w_{1,0}$ outperform the mean-variance efficient portfolio $w_{mv,0}$. For $u = 0$ the relevant hypothesis test is $H_{0,0,0} : LPM_{0,0}^{w_{0,0}}(\tau) \leq LPM_{0,0}^{w_{mv,0}}(\tau)$ for all $\tau \in (-\infty, 0]$, with critical value at 5% given by 1.572, and test-statistic 0.378. For $u = 1$ and $H_{0,0,1}$ the corresponding critical value is 1.558 and the test statistic 0.7621. The likelihood of each comovements event is 0.19 for $u = 0$ and 0.51 for $u = 1$.

$H_{0,0,0}: A \leq B$	$LPM_{0,0}^P$	$LPM_{1,0}^P$	$\sigma_{P,0}^2$
$A: LPM_{0,0}^P$	-	0	0
$A: LPM_{1,0}^P$	0	-	0
$A: \sigma_{P,0}^2$	1	1	-

$H_{0,0,1}: A \leq B$	$LPM_{0,1}^P$	$LPM_{1,1}^P$	$\sigma_{P,1}^2$
$A: LPM_{0,1}^P$	-	0	0
$A: LPM_{1,1}^P$	0	-	0
$A: \sigma_{P,1}^2$	1	1	-

Table 5.4. *Empirical rejection of the tests $H_{0,0,0}$ (left panel) and $H_{0,0,1}$ (right panel). 0 denotes acceptance of the relevant hypothesis and 1, rejection of the hypothesis at 5% significance level.*

We can conclude from table 5.4 that under comovements the downside-risk efficient portfolio dominates stochastically the mean-variance efficient portfolio for risk-neutral as well as for risk-averse investors. There is no need then to test for higher orders of stochastic dominance in this example since, as shown in Section 2, conditional first stochastic dominance implies the other orders of dominance.

6 Conclusions

The underlying risk in an investment portfolio can be decomposed into two components. A first element that measures the extent of interdependence or comovements between the assets in the portfolio, and a second element determined by the risk underlying the portfolio conditional on being or not under comovements episodes of the market. The risk in the latter component is determined by the choice of weights defining the portfolio. Therefore, we show that optimal investment portfolio decisions under uncertainty consist of two steps: a choice of assets comprising the portfolio aiming to minimize the likelihood of comovements, and a rule to determine the optimal share of investment on each asset.

One of the main implications of this decomposition of risk is that mean-risk and mean-variance efficient portfolios can be outperformed by portfolios with different weights in distress episodes of the market. To be able to decide whether this is possible or not in an investment environment we have introduced the concept of conditional stochastic dominance and provided some rules to derive optimal portfolios in crises episodes of the market. In order to make these rules statistically testable we have derived a testing framework for conditional stochastic dominance that can be adapted to any order of dominance and allows for some structures of dependence between portfolios, therefore shedding light about the preferences of investors with different degrees of risk-aversion.

In the empirical application we have carried out two experiments to gauge the importance of the mean-variance efficiency frontier in the context of stochastic dominance and under the presence of comovements between the assets in the portfolio. We can conclude from the data we have collected that mean-variance portfolios comprising assets already diversified, as financial indexes, are very likely to be second order stochastic dominant and hence the efficient portfolios for any type of risk-averse investor. Our experiment using data from US and UK also shows that the presence of comovements does not make investors switch to other mean-risk frontiers. On the contrary, we have observed that under comovements also risk-neutral investors become mean-variance agents. In the second experiment, on the other hand, with data from highly traded US stocks we have observed that mean-variance agents are myopic to the presence of comovements. In other words under distress periods of the market the optimal portfolios in an stochastic dominance sense are those obtained from the LPM_0-mean-risk model, being these portfolios the optimal choice of any risk-neutral and risk-averse investor.

Mathematical appendix

Proof of theorem 1: The proof of this theorem is immediate by applying the conditional probability theorem for $LPM_0(\tau)$ at a fixed value u, and using the results in equations (3) and (4). $\square$

Proof of proposition 1: The proof follows from observing that

$$E[U(x;q,\tau),F^i] := \int_{-\infty}^{u} x dF_u^i(x) - k \int_{-\infty}^{\tau} (x-\tau)^q dF_u^i(x) = \mu_u(i) - k\, LPM_{q,u}^i(\tau), \qquad (29)$$

with $i = A, B$, and $U(x;q,\tau)$ the utility function in (1). Therefore if A FCSD B then $\mu_u(A) > \mu_u(B)$ and $LPM_{q,u}^A(\tau) \leq LPM_{q,u}^B(\tau)$ for all $q \geq 0$. In the same way if A SCSD B then $\mu_u(A) \geq \mu_u(B)$ and $LPM_{q,u}^A(\tau) \leq LPM_{q,u}^B(\tau)$ for all $q \geq 1$, since for $q = 0$ the utility function is not concave and does not satisfy then the constraints on the real function v. The proof for TCSD involves $-\delta v/\delta x$ and is analogue to the two previous cases. $\square$

Proof of proposition 2: For $q = 0$ the formulas are immediately derived from Pickands, Balkema-de Haan theorem. Now, for $q = 1$, $LPM_1(\tau) = (\tau - E[R_p|R_p \leq \tau])LPM_0(\tau)$ by definition. Because τ is assumed to be in the left tail of the distribution of R_p we can approximate the conditional expected value using extreme value theory. It is standard in this theory to work with the right tail of the distribution, hence the rationale to study $E[-R_p| - R_p > -\tau]$ instead of $E[R_p|R_p \leq \tau]$). Thus, the conditional distribution of the upper tail converges to a Generalized Pareto distribution (GPD) as $-\tau$ goes to the right end point of the distribution. This result is the Pickands (1975), Balkema-de Haan (1974) theorem. The GPD takes the form

$$GPD_{\xi,\sigma_{p,-\tau}}(z) = \begin{cases} 1 - e^{\frac{-z}{\sigma_{p,-\tau}}} & \text{if } \xi_p = 0, \\ 1 - \left(1 + \xi_p \frac{z}{\sigma_{p,-\tau}}\right)^{-\frac{1}{\xi_p}} & \text{if } \xi_p \neq 0, \end{cases}$$

with $z = -R_p + \tau$. Then using that $E[-R_p| - R_p > -\tau] = -\tau + E[z|z > 0]$ we have the following:

$$E[-R_p| - R_p > -\tau] = \begin{cases} -\tau + \sigma_{p,-\tau}, & if \quad \xi_p = 0, \\ -\tau + \frac{\sigma_{p,-\tau}}{1-\xi_p}, & if \quad \xi_p \neq 0. \end{cases}$$

Note that $\sigma_{p,-\tau} = \sigma_{p,\tau}$ by construction, and therefore

$$E[R_p|R_p \leq \tau] = \begin{cases} \tau - \sigma_{p,\tau}, & if \quad \xi_p = 0, \\ \tau - \frac{\sigma_{p,\tau}}{1-\xi_p}, & if \quad \xi_p \neq 0. \end{cases}$$

Replacing in the expression for the LPM_1 measure and using the corresponding expressions for LPM_0 we obtain the postulated results. $\square$

For $q = 2$, we know that $LPM_2 = E[(R_p - \tau)^2 | R_p \leq \tau] LPM_0(\tau)$ by definition. Now to study the conditional expected value of $(R_p - \tau)^2$ in the tail we use the same arguments as in the preceding proof. Therefore we need to compute the approximation given by extreme value theory. Using integration by parts we obtain that

$$E[(R_p - \tau)^2 | R_p \leq \tau] = \begin{cases} 2\sigma_{p,\tau}^2, & if \quad \xi_p = 0, \\ \frac{2\sigma_{p,\tau}^2}{(1-\xi_p)(1-2\xi_p)}, & if \quad \xi_p \neq 0. \end{cases}$$

Replacing in the expression for the LPM_2 measure and using the corresponding expressions for LPM_0 we obtain the postulated results. $\square$

Proof of proposition 3: Suppose we have n independent and identically distributed vectors of observations from a random variable R, and let $\widehat{LPM}_\gamma(\tau)$ be the estimator of $LPM_\gamma(\tau)$ introduced in (10). This estimator takes this form:

$$\widehat{LPM}_\gamma(\tau) = \left(\frac{1}{n_p} \sum_{i=1}^{n} (\tau - x_i)^\gamma I(x_i \leq \tau) \right) \left(\frac{1}{n} \sum_{i=1}^{n} I(x_i \leq \tau) \right).$$

The expected value of this estimator is the following:

$$E\left[\widehat{LPM}_\gamma(\tau)\right] = E\left[E\left[\left(\frac{1}{n_p} \sum_{i=1}^{n} (\tau - x_i) I(x_i \leq \tau) \right) \left(\frac{1}{n} \sum_{i=1}^{n} I(x_i \leq \tau) \right) | X \leq \tau\right]\right].$$

By the law of iterated expectations

$$E[\widehat{LPM}_\gamma(\tau)] = E[(\tau - X)^\gamma | X \leq \tau] E[\left(\frac{1}{n_p} \sum_{i=1}^{n} I(x_i \leq \tau) \right)] = E[(\tau - X)^\gamma | X \leq \tau] F(\tau),$$

with $F(\tau)$ the distribution function of the random variable R. Note that $LPM_0(\tau) := F(\tau)$ and therefore by (3) we obtain the unbiasedness of the estimator.

The proof of the variance is similar but more tedious. By definition we know that

$$V(\widehat{LPM}_\gamma(\tau)) = E[\widehat{LPM}_\gamma^2(\tau)] - E^2[(\tau - X)^\gamma | X \leq \tau] F^2(\tau).$$

After some algebra and applying the law of iterated expectations to the first term on the right of the expression we obtain

$$E[\widehat{LPM}_\gamma^2(\tau)] =$$
$$E\left[\frac{1}{n^2} \sum_{i=1}^{n} I(x_i \leq \tau) + \frac{1}{n^2} \sum_{i=1} \sum_{\substack{j=1 \\ j \neq i}}^{n} I(x_i \leq \tau) I(x_j \leq \tau)\right] E\left[\frac{1}{n_p^2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\tau - x_i)^\gamma (\tau - x_j)^\gamma | X \leq \tau\right].$$

By the independence between the observations and after some algebra the former expression reads as

$$E[\widehat{LPM}_\gamma^2(\tau)] =$$
$$\left(\tfrac{1}{n}F(\tau)(1-F(\tau))+F(\tau)^2\right)\left(\tfrac{1}{n_p}E[(\tau-X)^{2\gamma}|X\leq\tau]+\tfrac{n_p-1}{n_p}E^2[(\tau-X)^{\gamma}|X\leq\tau]\right).$$

By summing and substracting $\frac{1}{n^p}E^2[(\tau-X)^{\gamma}|X\leq\tau]$ on the right term we obtain that

$$E[\widehat{LPM}_\gamma^2(\tau)] = \left(\tfrac{1}{n}F(\tau)(1-F(\tau))+F(\tau)^2\right)\left(\tfrac{1}{n^p}V[(\tau-X)^{2\gamma}|X\leq\tau]+E^2[(\tau-X)^{\gamma}|X\leq\tau]\right).$$

Then

$$V[\widehat{LPM}_\gamma^2(\tau)] =$$
$$\tfrac{1}{n}F(\tau)(1-F(\tau))\left(\tfrac{1}{n_p}V[(\tau-X)^{\gamma}|X\leq\tau]+E^2[(\tau-X)^{\gamma}|X\leq\tau]\right)+\tfrac{1}{n_p}V[(\tau-X)^{\gamma}|X\leq\tau]F(\tau)^2.$$

Note that this term converges to zero as n and n_p tend to infinity, proving the consistency of the nonparametric estimators of the $LPM_\gamma(\tau)$ measures introduced before. Therefore, as n_u and n increases we have by the central limit theorem that

$$\sqrt{n}\frac{\widehat{LPM}_\gamma(\tau)-LPM_\gamma(\tau)}{\sqrt{F(\tau)(1-F(\tau))\left(\frac{1}{n_p}V[(\tau-X)^{2\gamma}|X\leq\tau]+E^2[(\tau-X)^{\gamma}|X\leq\tau]\right)}}\xrightarrow{d}N(0,1). \quad (30)$$

Hence

$$\sqrt{n}\left(\widehat{LPM}_\gamma(\tau)-LPM_\gamma(\tau)\right)\xrightarrow{d}N\left(0,F(\tau)(1-F(\tau))E^2[(\tau-X)^{\gamma}|X\leq\tau]\right). \quad (31)$$

□

Proof of proposition 4: This result immediate follows from the unbiasedness of the LPM_γ estimators, for $\gamma\geq 0$, and the multivariate central limit theorem. The proof is similar to the proof of proposition 4 and therefore is omitted for sake of space. □

Proof of proposition 5: We need the asymptotic result in (22) and the tightness of the empirical process as a random function of τ. Since the class of functions we are interested in belongs to the Donsker class, see van der Vaart (1998, chapter 19), the proof of the theorem follows from Donsker theorem. □

Proof of theorem 2: Under the null hypothesis $H_{0,0}$: $LPM_\gamma^A(\tau)\leq LPM_\gamma^B(\tau)$ for all $\tau\in\mathbb{R}$. We consider the frontier case given by $LPM_\gamma^A(\tau)\leq LPM_\gamma^B(\tau)$ to represent the null hypothesis. Then, it is immediate to derive the null distribution of $T_n(\gamma)$ that is characterized by the covariance function obtained from replacing $k_\gamma^A(\tau)$ and $F^A(\tau)$ by $k_\gamma^B(\tau)$ and $F^B(\tau)$. □

Proof of theorem 3: It is analogous to the proof of theorem 2 but replacing the relevant unconditional distribution functions by their conditional counterparts, with the conditioning event defined by a threshold u. □

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